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ON THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH SUCTION†

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Abstract—The compressible laminar boundary layer in a pressure gradient with suction is analyzed on the basis of the momentum and thermal integral equations in conjunction with sixth and (for separation) seventh degree velocity, and seventh degree stagnation enthalpy profiles. For flow over a flat plate and for flows in a pressure gradient, straightforward and simple methods of calculating the boundary layer for a given Mach number, a given uniform wall temperature and a given suction distribution are shown. The results obtained by the present analysis agree well with available exact or purportedly accurate solutions for an impermeable or permeable wall, including the general asymptotic suction solutions for compressible flows with a pressure gradient and heat transfer. The asymptotic solutions imply that it should always be possible to entirely prevent separation in a given (finite) adverse pressure gradient by sufficient suction. A mathematically simple class of solutions in which the pressure gradient is arbitrarily prescribed, but the suction distribution is implicitly determined, is shown. Finally, the boundary layer with a linearly diminishing external velocity is calculated in detail. Of especial interest here is the delay and complete prevention of separation by suction, including determination of the minimum (homogeneous) suction parameter to entirely avoid separation. Effects of Mach number and wall temperature are shown.

NOMENCLATURE

A, defined by equation (13e);

a_i, coefficient of η^i in velocity profiles;

B, $(1/K)\,h'(\rho_1/\rho_\infty)(T_\infty/T_1)(u_1/u_\infty)\lambda$;

b_i, coefficient of η^i in stagnation enthalpy profiles;

$\bar{b}_1$, approximation for $b_1(\xi)$, especially as given by equation (37b);

c, $(1/K)(T_\infty/T_1)\ \varphi\sqrt{\lambda}$ (positive for suction);

c_f, local skin friction coefficient [see equation (19)];

c_p, specific heat at constant pressure;

F_1, $\int_0^1 (u/u_1)[1-(u/u_1)]\,\mathrm{d}\eta$;

F_2, $\int_0^1 [(H/H_1)-(u/u_1)^2]\,\mathrm{d}\eta$;

F_3, $[\partial/\partial\eta(u/u_1)]_w$;

F_4, $\int_0^1 (u/u_1)[1-(H/H_1)]\,\mathrm{d}\eta$;

F_5, $[\partial/\partial\eta(H/H_1)]_w$;

H, stagnation enthalpy $[(u^2/2)+c_pT]$;

h, H_w/H_1
$= (T_w/T_1)/\{1+[(\gamma-1)/2]M_1^2\}$
$= (T_w/T_\infty)/\{1+[(\gamma-1)/2]M_\infty^2\}$.
For $Pr=1$, $h=T_w/T_e$;

K, defined by equations (5a) and (5b);

k, coefficient of heat conductivity;

L, streamwise characteristic length;

M, Mach number;

m, $1+[(\gamma-1)/2]M_\infty^2$;

Pr, Prandtl number $(\mu c_p/k)$;

Re, $(\rho_\infty u_\infty L/\mu_\infty)$ Reynolds number;

R_x, $\rho_\infty u_\infty x/\mu_\infty$;

r, $c[(2b_1/h)-c]$;

S, Sutherland constant (216°R for air);

T, absolute temperature;

T_e, equilibrium wall temperature for

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zero heat transfer. For $Pr = 1$,
$T_e = T_\infty\{1 + [(\gamma - 1)/2]M_\infty^2\}$;

t, variable defined by equation (7);
u, v, velocity components in x and y directions, respectively;
x, y, coordinates along and normal to wall, respectively;
z, c^2.

Greek symbols

α_1, α_2, defined by equations (53) and (56), respectively;
β_1, β_2, defined by equations (53) and (56), respectively;
γ, ratio of specific heats (c_p/c_v). ($\gamma = 1{\cdot}4$ for air);
δ, boundary-layer thickness;
δ_t, boundary-layer thickness in (x, t) plane;
δ^*, displacement thickness [see equations (22)];
θ, momentum thickness [see equation (21)];
η, t/δ_t;
λ, $(\delta_t/L)^2\, Re$;
μ, coefficient of viscosity;
ξ, x/L;
ζ, $-\rho_w v_w \int_0^y (\mathrm{d}y/\mu)$ (reference [24]);
ρ, mass density;
σ, $\varphi(\sqrt{2\xi})$;
σ_1, $(-v_w/u_\infty)R_x^{\frac{1}{2}}$;
Φ_1, Φ_2, defined by equations (53) and (56), respectively;
φ, $-(\rho_w v_w/\rho_\infty u_\infty)\, Re^{\frac{1}{2}}$ (positive for suction, negative for injection).

Subscripts

1, value at local outer edge of boundary layer;
∞, value at reference point outside of boundary layer;
s, value at the separation point;
w, value at wall.

A prime (′) denotes derivative with respect to ξ.
A bar ($\bar{\ }$) denotes a constant ("average") value for flow (except for $\bar{b}_1$).

1. INTRODUCTION

THE USE of suction for boundary-layer control, for example in prevention of separation and in stabilizing the laminar boundary layer, has been of considerable practical and theoretical interest for some time [1]. The purpose of the present paper is to present a theoretical analysis of the compressible laminar boundary layer in a pressure gradient with suction, including heat transfer. For simplicity, a Prandtl number of unity and a linear viscosity temperature relation are assumed. The analysis is based on the use of the momentum and thermal integral equations of the laminar boundary layer in conjunction with sixth-degree velocity profiles and seventh-degree stagnation enthalpy profiles. For the special purpose of locating still more accurately the separation point in an adverse pressure gradient, seventh-degree velocity profiles satisfying an additional condition at the separation point are used. The present method is an application to suction of the method previously used for impermeable walls [2] and for fluid injection [3]. Momentum integral methods for the laminar boundary layer with suction have, of course, been used previously [4–8]. In many cases, however, they have either been restricted to flows without a pressure gradient, have been applied only to rather specific problems and remained of unknown accuracy, have not been sufficiently accurate for predicting the separation point, or have broken down when the suction reached a certain magnitude. The method used here has the following properties. It is entirely self-contained, not making use of any known exact solutions; it has been found to agree well with exact or purportedly accurate incompressible or compressible solutions for prediction of the separation point over an impermeable wall in a variety of adverse pressure gradients [9]; it agrees well with exact solutions for the flow over a flat plate with uniform suction or with $v_w \sim x^{-\frac{1}{2}}$; for sufficiently large suction it yields solutions in agreement with the exact general asymptotic suction solutions for compressible flows with or without a pressure gradient. The method is fairly general, applying to flows in a given pressure gradient, a given Mach number, a given suction distribution at the wall, and (in case of heat transfer) a given uniform wall temperature. Profiles can be obtained here, as well as any other properties. The method is fairly straightforward to apply, involving at most the solution of a first-order ordinary differential equation. Moreover, a simplified approximate method of solving this latter equation is shown here. (Since, however, the method is of the one-parameter type, it may perhaps not be adequate in dealing with initial-value problems in which, for example, the velocity profile is specified at a point downstream of the leading edge.)

A survey of literature on the laminar boundary layer with suction has been made by Lew and Mathieu [10] as of 1954, and more recently by Wuest [11]. Concerning approximate methods of solution for incompressible flow, in addition to the momentum integral methods indicated above, four recent contributions may be mentioned. Head [12–14] has developed a method based on the use of both a momentum and energy integral equation of the laminar boundary layer in conjunction with a doubly infinite family of velocity profiles chosen in accordance with various types of exact solutions. Curle [15] has extended Stratford's method to suction, to determine skin friction and separation. Smith and Clutter [16] have recently applied a finite-difference method of the Hartree–Wormersley type. Spalding [40] has developed a modification of the Pohlhausen method along the lines of Walz and Thwaites, using known similarity solutions.

In addition to Iglisch's solution [17] for the incompressible flow over a flat plate with homogeneous suction, there are mostly two types of known exact solutions of the laminar boundary-layer equations with suction, namely the asymptotic suction solutions and the similarity solutions. For the limit of very large suction, the well-known asymptotic suction velocity profile for uniform suction and zero pressure gradient was first found by Griffith and Meredith [18] and (independently) by Schlichting [19]. It was subsequently shown by Pretsch [20] and (independently) by Watson [21] that this profile holds essentially also with pressure gradients and variable suction velocity, in incompressible flow. Young [22] obtained the asymptotic suction solution for compressible flow with zero heat transfer for flow without a pressure gradient and uniform suction. Lew and Fanucci [23] subsequently extended this analysis to compressible flow over a flat plate with a given uniform wall temperature. Morduchow [24] has recently derived the asymptotic suction velocity and temperature profiles for compressible flows in a pressure gradient with heat transfer under quite general conditions. These latter solutions for u/u_1 and T/T_1, which will be discussed more fully in the main text, are those which will be approached by the solution of the laminar boundary-layer equations in *any* pressure gradient with any given suction velocity and wall temperature distributions.

The similarity solutions are those in which the pressure gradient, suction distribution and wall temperature distribution are such that the partial differential equations of the laminar boundary layer can be converted exactly into ordinary differential equations. A number of investigators have considered such solutions, but it will suffice here to note the solutions for the incompressible flow over a flat plate with $v_w \sim x^{-\frac{1}{2}}$ first obtained by Schlichting and Bussmann [25] and (independently) by Thwaites [26] and Emmons and Leigh [27], and the recent numerical solutions obtained by Koh and Hartnett [28] for low-speed flows in which $u_1 \sim x^m, v_w(x) \sim x^{(m-1)/2}, T_w(x) - T_1 \sim x^n, Pr = 0{\cdot}73$ and $m = 0, \frac{1}{3}, 1$; $n = -1$ to 10.†

Most of the literature (with which no attempt has been made here to deal comprehensively) on the boundary layer with suction has concerned incompressible flow. Regarding com-

† Quite recently Zamir and Young [41] have extended these latter type of solutions to various *negative* values of m, for incompressible flow.

pressible flow, Lew and Fanucci [23] obtained a solution for the compressible flow over a flat plate with uniform suction (or injection) and uniform wall temperature. Assuming $\mu \sim T$ and $Pr = 1$, they showed that the equations could be transformed into those for incompressible flow. The asymptotic suction solutions for compressible flows have already been noted [22–24]. Flügge-Lotz and Howe [29] have applied a finite-difference method for calculations of the compressible laminar boundary layer with suction (or injection) over hot or cold walls with and without a pressure gradient. Gribben [30] has shown that for $Pr = 1$ and $\mu \sim T$, the axisymmetric equations for a given compressible flow with zero heat transfer and suction or injection can be transformed into an incompressible two-dimensional flow with a determinable main stream and normal wall velocity distribution. He also showed that with heat transfer, a high-speed flow can be correlated with a low-speed compressible flow. Pechau [31, 32] has recently extended Schlichting's integral method [5] in conjunction with an approximation of Truckenbrodt, to compressible flows with zero heat transfer.

In the present study, in addition to presenting a method of analysis of the compressible laminar boundary layer with suction, the results for flow with a linearly diminishing external velocity are given in detail, including separation, minimum ("critical") uniform suction to prevent separation entirely, approach of the solutions to the asymptotic suction profiles for large suction, and Mach number and heat-transfer effects. Also included in the present analysis is a class of mathematically relativity simple ("constant-c") solutions in which the pressure gradient may be arbitrarily prescribed, but the suction distribution then becomes automatically determined, and which include as a special case the similarity flow over a flat plate with $v_w \sim x^{-\frac{1}{2}}$.

2. BASIC EQUATIONS

The continuity, momentum and thermal boundary-layer equations for compressible, steady, two-dimensional flow are [33]:

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \qquad (1)$$

$$\rho u \partial u/\partial x + \rho v \partial u/\partial y = \rho_1 u_1 \, \mathrm{d}u_1/\mathrm{d}x + \partial/\partial y(\mu \partial u/\partial y) \qquad (2)$$

$$\rho u c_p \partial T/\partial x + \rho v c_p \partial T/\partial y = \partial/\partial y(k \partial T/\partial y) + \mu(\partial u/\partial y)^2 - \rho_1 u_1 (\mathrm{d}u_1/\mathrm{d}x) u. \qquad (3)$$

From the ideal gas law, together with the boundary-layer equation $\partial p/\partial y = 0$, it follows that

$$\rho/\rho_1 = T_1/T. \qquad (4)$$

The coefficients of specific heat, and the Prandtl number Pr, will be assumed as constant, with $Pr = 1$ eventually, while the viscosity coefficient μ will be assumed as proportional to the absolute temperature in the form [34]

$$\mu/\mu_\infty = K(T/T_\infty) \qquad (5a)$$

where K is chosen to satisfy the Sutherland relation at the wall:

$$K = [(T_\infty + S)/(T_w + S)]\,(T_w/T_\infty)^{\frac{1}{2}}. \qquad (5b)$$

In this analysis the wall temperature T_w will be assumed uniform; hence K will be constant. For a constant Prandtl number, equation (3) may be replaced by the equation [2]:

$$Pr[\rho u \partial H/\partial x + \rho v \partial H/\partial y] = \partial/\partial y(\mu \partial H/\partial y) - (1 - Pr)\,\partial/\partial y(\mu u \partial u/\partial y). \qquad (6)$$

It will be convenient to introduce the (Dorodnitsyn) variable t defined by

$$y = \int_0^t (T/T_1)\,\mathrm{d}t. \qquad (7)$$

If a common momentum and thermal boundary-layer thickness is assumed, equations (2) and (6), in conjunction with equations (1), (5a) and (7) may be integrated with respect to y over the boundary-layer thickness to yield [3]:

$$\tfrac{1}{2}F_1\lambda' + \left\{F_1\frac{\rho_1'}{\rho_1} + F_1' + \left[F_1 + \left(1 + \frac{\gamma - 1}{2}M_1^2\right)F_2 \quad \frac{u_1'}{u_1}\right\}\lambda = K\frac{u_\infty}{u_1}\frac{\rho_\infty}{\rho_1}\frac{T_1}{T_\infty}(F_3 - c) \qquad (8)$$

$$\tfrac{1}{2}F_4\lambda' + \left[F_4' + \left(\frac{\rho_1'}{\rho_1} + \frac{u_1'}{u_1}\right)F_4\right]\lambda = \frac{K}{Pr}\frac{T_1}{T_\infty}\frac{\rho_\infty}{\rho_1}\frac{u_\infty}{u_1}[F_5 - Pr(1-h)c] \quad (9)$$

where

$$c = (1/K)(T_\infty/T_1)\varphi\sqrt{\lambda}. \quad (10)$$

An approximate solution to the basic set of partial differential equations can now be obtained by assuming the velocity and stagnation enthalpy as definite functions of t, with certain parameters as unknown functions of ξ required to satisfy equations (8) and (9). In this analysis, sixth degree velocity and seventh degree stagnation enthalpy profiles will be chosen to satisfy the following boundary conditions:

At $\eta = 1$:

$$u/u_1 = 1, \quad H/H_1 = 1 \quad (11a)$$

$$\partial(u/u_1)/\partial\eta = \partial^2(u/u_1)/\partial\eta^2 = \partial^3(u/u_1)/\partial\eta^3 = 0 \quad (11b)$$

$$\partial(H/H_1)/\partial\eta = \partial^2(H/H_1)/\partial\eta^2 = \partial^3(H/H_1)/\partial\eta^3 = 0. \quad (11c)$$

At $\eta = 0$:

$$u/u_1 = 0, \qquad H/H_1 = h \quad (11d)$$

$$-c\partial(u/u_1)/\partial\eta = A + \partial^2(u/u_1)/\partial\eta^2 \quad (11e)$$

$$-ch\partial^2(u/u_1)/\partial\eta^2 = h\partial^3(u/u_1)/\partial\eta^3 + A\partial(H/H_1)/\partial\eta \quad (11f)$$

$$-c\partial(H/H_1)/\partial\eta = (1/Pr)\partial^2(H/H_1)/\partial\eta^2 \quad (11g)$$

$$B\partial(u/u_1)/\partial\eta - c\partial^2(H/H_1)/\partial\eta^2 = (1/Pr)\,\partial^3(H/H_1)/\partial\eta^3 - [(1-Pr)/Pr]\,(u_1^2/H_1)\,\{3\partial(u/u_1)/\partial\eta \times \partial^2(u/u_1)/\partial\eta^2 - (1/h)\,[\partial(H/H_1)/\partial\eta] \times [\partial(u/u_1)/\partial\eta]^2\}. \quad (11h)$$

Equations (11e) and (11g) follow, respectively, from equations (2) and (6) evaluated at the wall, with $v = v_w(x)$ there, while equations (11f) and (11h) follow by differentiating equations (2) and (6) with respect to η and evaluating the resulting terms at the wall (cf. [2, 3, 35]).

For the case of $Pr = 1$ and uniform wall temperature ($h' = 0$) the seventh degree stagnation enthalpy profiles and the sixth degree velocity profiles satisfying the above conditions are:

$$H/H_1 = h + (1-h)(35\eta^4 - 84\eta^5 + 70\eta^6 - 20\eta^7) + b_1(\eta - 20\eta^4 + 45\eta^5 - 36\eta^6 + 10\eta^7) - (c/2)\,b_1(\eta^2 - 10\eta^4 + 20\eta^5 - 15\eta^6 + 4\eta^7) + (c^2/6)\,b_1(\eta^3 - 4\eta^4 + 6\eta^5 - 4\eta^6 + \eta^7) \quad (12)$$

$$u/u_1 = \sum_{i=1}^{6} a_i\eta^i \quad (13a)$$

where

$$a_1 = 2 - \frac{1}{5}\left(2 - \frac{c}{6}\right)a_2 + \frac{Ab_1}{60h} = \{120 + A[12 - c + (b_1/h)]\}/(c^2 - 12c + 60) \quad (13b)$$

$$a_3 = -\frac{1}{3}ca_2 - \frac{Ab_1}{6h};$$

$$a_4 = -5 + 2\left(\frac{c}{3} - 1\right)a_2 + \frac{Ab_1}{3h}$$

$$a_5 = 6 + \left(2 - \frac{c}{2}\right)a_2 - \frac{Ab_1}{4h}; \quad (13c)$$

$$a_6 = -2 - \frac{1}{5}\left(3 - \frac{2}{3}c\right)a_2 + \frac{Ab_1}{15h}$$

$$a_2 = -\frac{60c + A[30 + (b_1c/2h)]}{c^2 - 12c + 60} \quad (13d)$$

$$A = (1/K)(T_\infty/T_1)(\rho_1/\rho_\infty)(T_w/T_1)(u_1'/u_\infty)\lambda. \quad (13e)$$

With these profiles, the explicit expressions for F_1 to F_5 are found to be:

$$F_1 = 0{\cdot}10934 + [0{\cdot}002109 - 0{\cdot}0004126c + 0{\cdot}0000764\,(Ab_1/h) - 0{\cdot}0000095\,(Ab_1/h)c]a_2 - [0{\cdot}0006216 - 0{\cdot}0001528c + 0{\cdot}0000095c^2]\,a_2^2 - [0{\cdot}0002063 + 0{\cdot}0000024\,(Ab_1/h)]\,(Ab_1/h) \quad (14a)$$

$$F_2 = F_1 + (\tfrac{1}{2})(1+h) - 0{\cdot}71429 + (0{\cdot}01905 - 0{\cdot}002381c)\,a_2 + (0{\cdot}10714 - 0{\cdot}01190c + 0{\cdot}000595c^2)\,b_1 - 0{\cdot}001190\,(Ab_1/h) \quad (14b)$$

$$F_3 = a_1; \qquad F_5 = b_1 \tag{14c}$$

$$\begin{aligned} F_4 = (1 - h)[0{\cdot}24603 - (0{\cdot}01496 \\ - 0{\cdot}001806c)a_2 + 0{\cdot}000903(Ab_1/h)] \\ - b_1[0{\cdot}06835 - (0{\cdot}003247 - 0{\cdot}000411c)a_2 \\ + 0{\cdot}000205(Ab_1/h)] - b_1c[0{\cdot}008381 \\ - (0{\cdot}000339 - 0{\cdot}000044c)a_2 \\ + 0{\cdot}000022(Ab_1/h)] - b_1c^2[0{\cdot}000441 \\ - (0{\cdot}000016 - 0{\cdot}0000021c)a_2 \\ + 0{\cdot}0000010(Ab_1/h)]. \end{aligned} \tag{14d}$$

With profiles and expressions for F_1 to F_5 such as the above, equations (8) and (9) become two ordinary differential equations in $\lambda(\xi)$ and $b_1(\xi)$. For a permeable wall with suction it is often convenient to express equations (8) and (9) with $c(\xi)$, instead of $\lambda(\xi)$, as an unknown. Thus,

$$\begin{aligned} cc' + c^2 \Bigg\{\frac{T_1'}{T_1} - \frac{\varphi'}{\varphi} + \frac{\rho_1'}{\rho_1} + \frac{F_1'}{F_1} \\ + \left[1 + \left\{1 + \left(\frac{\gamma - 1}{2}\right)M_1^2\right\}\frac{F_2}{F_1}\right]\frac{u_1'}{u_1}\Bigg\} \\ = \frac{1}{K}\frac{T_\infty}{T_1}\frac{\rho_\infty}{\rho_1}\frac{u_\infty}{u_1}\varphi^2\frac{(a_1 - c)}{F_1} \end{aligned} \tag{15}$$

$$\begin{aligned} cc' + c^2\left\{\frac{T_1'}{T_1} - \frac{\varphi'}{\varphi} + \frac{\rho_1'}{\rho_1} + \frac{F_4'}{F_4} + \frac{u_1'}{u_1}\right\} \\ = \frac{1}{K}\frac{T_\infty}{T_1}\frac{\rho_\infty}{\rho_1}\frac{u_\infty}{u_1}\varphi^2\frac{[b_1 - (1 - h)c]}{F_4}. \end{aligned} \tag{16}$$

For zero heat transfer at the wall, the solution of the energy equation (6), for $Pr = 1$, is $H =$ constant, whence the following well-known relation between the temperature and velocity is obtained for this case:

$$T/T_1 = 1 + [(\gamma - 1)/2]M_1^2[1 - (u/u_1)^2]. \tag{17}$$

From equation (17) it will follow that in this case $h = 1$. For the more general case of heat transfer, the temperature profiles can be obtained from the stagnation enthalpy and velocity profiles by means of the relation

$$\begin{aligned} T/T_1 = (H/H_1)\{1 + [(\gamma - 1)/2]M_1^2\} \\ - (u/u_1)^2[(\gamma - 1)/2]M_1^2. \end{aligned} \tag{18}$$

After the differential equations have been solved, the desired boundary-layer properties can be straightforwardly calculated. For example, the local skin friction coefficient, Nusselt number, momentum thickness and displacement thickness will be, respectively:

$$c_f \equiv \frac{(\mu\partial u/\partial y)_w}{(\frac{1}{2})\rho_\infty u_\infty^2} = 2\frac{u_1}{u_\infty}\frac{a_1\varphi}{c}Re^{-\frac{1}{2}} \tag{19}$$

$$Nu \equiv \frac{(k\partial T/\partial y)_w L}{k_\infty(T_e - T_w)} = \frac{b_1\varphi}{(1 - h)c}Re^{\frac{1}{2}} \tag{20}$$

$$\theta \equiv \int_0^\delta \frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right)dy = K\frac{c}{\varphi}\frac{T_1}{T_\infty}F_1\,Re^{-\frac{1}{2}}L \tag{21}$$

$$\begin{aligned} \delta^* \equiv \int_0^\delta \left(1 - \frac{\rho u}{\rho_1 u_1}\right)dy \\ = K\frac{c}{\varphi}\left[mF_2 - \frac{T_1}{T_\infty}F_1\right]Re^{-\frac{1}{2}}L. \end{aligned} \tag{22}$$

From equations (19–22),

$$(Nu/c_f R_e) = [b_1/2(1 - h)a_1](u_\infty/u_1) \tag{23}$$

$$\delta^*/\theta = (F_2/F_1)m(T_\infty/T_1) - 1. \tag{24}$$

According to equation (7) the transformation back to the physical (x, y) plane can be made by

$$(y/L)Re^{\frac{1}{2}} = \sqrt{\lambda}\int_0^\eta (T/T_1)\,d\eta. \tag{25}$$

For the (isentropic) flow at the local outer edge of the boundary layer, the following relations hold:

$$\begin{aligned} T_1/T_\infty = 1 + [(\gamma - 1)/2]M_\infty^2[1 - (u_1/u_\infty)^2] \\ \rho_1/\rho_\infty = (T_1/T_\infty)^{1/(\gamma - 1)}; \\ M_1^2 = M_\infty^2(u_1/u_\infty)^2(T_1/T_\infty)^{-1}. \end{aligned} \tag{26a}$$

From these relations, the following useful relation may be readily derived:

$$\begin{aligned} 1 + [(\gamma - 1)/2]M_1^2 = \{1 + [(\gamma - 1)/2]M_\infty^2\} \\ \times (T_\infty/T_1). \end{aligned} \tag{26b}$$

3. FLOW OVER A FLAT PLATE

For the special case of zero pressure gradient (such as flow over a flat plate at zero angle of attack), in which $u_1/u_\infty \equiv T_1/T_\infty \equiv 1$, $u_1' \equiv 0$, $A \equiv 0$, the foregoing equations simplify considerably. First, the results obtained by the equations given here will be compared with certain exact (numerical) solutions. Then it will be shown that the solutions for compressible flow with heat transfer can be readily obtained here from the solutions for incompressible flow. (This is also in accord with exact analyses.) Finally, a comparatively simple method will be given for calculating the boundary layer over a flat plate with a prescribed distribution of normal velocity $v_w(x)$ along the wall and a prescribed uniform wall temperature.

For flow over a flat plate, since $A = 0$, equation (13b) reduces to:

$$a_1 = 120/(c^2 - 12c + 60). \tag{27}$$

Case of homogeneous suction

For incompressible flow over a flat plate with $v_w =$ constant, equation (15) can be reduced to

$$(cF_1 + c^2\,\mathrm{d}F_1/\mathrm{d}c)c' = \varphi^2(a_1 - c) \tag{28}$$

where φ is constant. Since with $A = 0$, F_1 will be a function of c only, equation (28) can be solved for ξ vs. c by separation of variables and a quadrature. However, as will be seen, the range of c here will be $0 \leqslant c \leqslant 4{\cdot}644$, and for this range, it will be found that F_1 is virtually constant. Hence the calculations can be simplified here still further by replacing $F_1(c)$ by a constant "average" value $\bar{F}_1$ for the flow. The solution of equation (28) in conjunction with equation (27), with $c = 0$ at $\xi = 0$, is then found to be:

$$\begin{aligned}\varphi^2\xi = \bar{F}_1\{&-c - 9{\cdot}060\ln[1 - (c/4{\cdot}644)] \\ &+ 4{\cdot}530\ln(1 - [c(7{\cdot}356 - c)/25{\cdot}84]) \\ &+ 2{\cdot}496\,(0{\cdot}8089 - \arctan[(7{\cdot}356 \\ &\qquad - 2c)/7{\cdot}018])\}.\end{aligned} \tag{29}$$

From equation (29), $2\varphi^2\xi \equiv \sigma^2$ can be readily calculated as a function of c. For the case $\sigma \to \infty$ (asymptotic suction solution) it is seen that $c = 4{\cdot}644$. Exact (numerical) solutions for

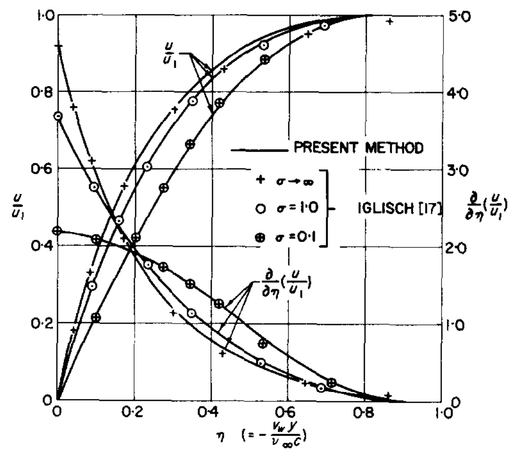

FIG. 1. Comparison of calculated profiles and their derivatives with exact (numerical) solutions. Incompressible flow over a flat plate with uniform suction.

the homogeneous suction case have been calculated by Iglisch [17]. The results of the equations developed here for the velocity profiles and their first derivatives (to which the local skin friction is proportional) are compared in Fig. 1 with the exact solution for $\sigma = 0{\cdot}1$, $1{\cdot}0$ and ∞, and the agreement is seen to be satisfactory throughout. The agreement with the exact asymptotic suction profile ($\sigma \to \infty$) is especially noteworthy in view of the fact that the velocity profiles were chosen here simply as (sixth degree) polynomials, without attempting to match them in advance with the known exact (exponential) asymptotic profile. Moreover, this agreement is particularly important since both the exact solutions [24] and, as will be seen subsequently, the approximate solutions developed here, approach essentially this same asymptotic solution when $v_w \to -\infty$ even in the presence of a pressure gradient and a variable $v_w(x)$.

Case of $v_w \sim x^{-\frac{1}{2}}$

Consider now the special case of incompressible flow along a flat plate in which $v_w(x)$ is such that c is constant along the flow. Then F_1

will also be constant, and equation (8) reduces to

$$\lambda' = (2/F_1)(a_1 - c) \tag{30}$$

where a_1 is given by equation (27). With the condition $\lambda = 0$ at $\xi = 0$, the solution of equation (30) is

$$\lambda = (2/F_1)(a_1 - c)\xi. \tag{31}$$

From equation (10),

$$c = \varphi\sqrt{\lambda}. \tag{32}$$

Eliminating λ from equations (31) and (32), it is found that

$$-v_w(x)/u_\infty = c\{\sqrt{F_1}/[2(a_1 - c)]\}R_x^{-\frac{1}{2}}. \tag{33}$$

Equation (33) shows that $v_w(x) \sim x^{-\frac{1}{2}}$. In this case, the original partial differential equations can actually be reduced exactly to an ordinary differential equation, and exact solutions for this case have been calculated [25–27]. The equations obtained here can be used to calculate velocity profiles, displacement and momentum thicknesses, and skin friction for any values of

$$\sigma_1 \equiv (-v_w/u_\infty)R_x^{\frac{1}{2}} = c\{\sqrt{F_1}/[2(a_1 - c)]\}. \tag{34}$$

Equation (34) yields c vs. σ_1. Figures 2 and 3 show the satisfactory agreement of the results calculated here with those of exact solutions for all values of σ_1 calculated.

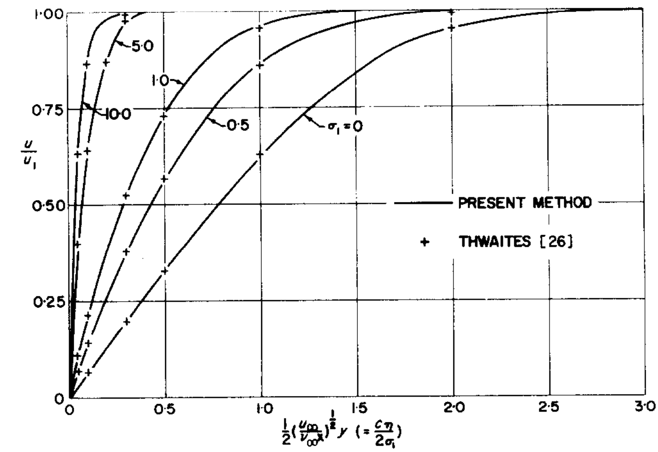

FIG. 2. Comparison of calculated velocity profiles with exact (numerical) solutions. Incompressible flow over a flat plate. Suction velocity $\sim x^{-\frac{1}{2}}$.

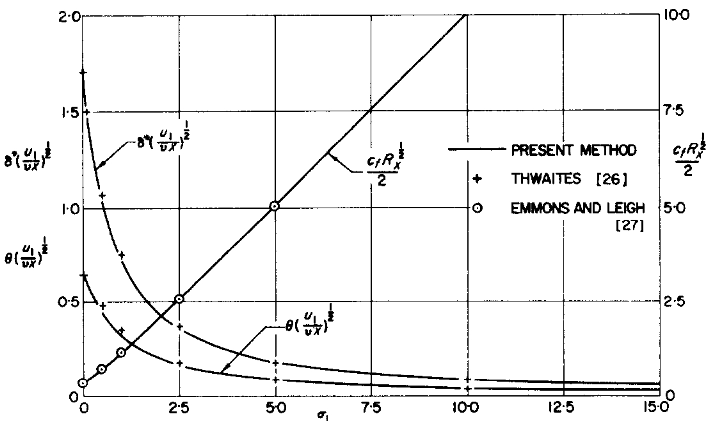

FIG. 3. Comparison of calculated distribution of c_f, δ^* and θ with exact (numerical) solutions. Incompressible flow over a flat plate with suction velocity $\sim x^{-\frac{1}{2}}$.

It should be noted that although exact (numerical) calculations for suction have been carried out only for $0 \leqslant \sigma_1 \leqslant 10$, the equations developed here can be used for the entire suction range $0 \leqslant \sigma_1 \leqslant \infty$. In fact, corresponding to $\sigma_1 \to \infty$, c is such that $a_1 - c = 0$, whence $c = 4{\cdot}644$. This is the same value of c as that corresponding to the asymptotic suction profile for the case of homogeneous suction. Moreover, for the present incompressible flows, equation (32) and the definitions of η, λ and φ imply that $\eta = (1/c)(-\rho_\infty v_w y/\mu_\infty)$. Consequently, since the a_i are here functions of c only and u/u_1 is given by equation (13a), the approximate solution developed here implies that in the asymptotic case ($\sigma_1 \to \infty$), the velocity profile u/u_1 for the case $v_w(x) \sim x^{-\frac{1}{2}}$ will be exactly the same function of $(-\rho_\infty v_w(x)y/\mu_\infty)$ as in the case of homogeneous suction. This is in exact agreement with the implications of the asymptotic suction profiles [20, 21, 24].

General solution for compressible flow with heat transfer and arbitrary $v_w(x)$

For the general case of compressible flow over a flat plate with a prescribed uniform wall temperature, equation (15) can be written in the form

$$(cF_1 + c^2\,\mathrm{d}F_1/\mathrm{d}c)c' = (\varphi^2/K)(a_1 - c) + F_1 c^2(\varphi'/\varphi) \qquad (35)$$

where F_1 and, in fact, all the coefficients a_i in the velocity profile, are functions of c only. Equation (35) for $c(\xi)$ is the same as the corresponding equation for incompressible flow with a mass flow parameter φ_i given by

$$\varphi_i(x) = \varphi(x)/\sqrt{K}. \qquad (36)$$

Thus $c(\xi)$ will be the same as for an incompressible flow with a mass flow distribution given by equation (36). Therefore the velocity profiles for compressible flow with a given uniform wall temperature T_w and a prescribed normal mass flow distribution $\varphi(x)$ will be the same functions of ξ and η as those for incompressible flow with a normal mass flow $\varphi(x)/\sqrt{K}$, where K is a constant, given by equation (5b).

For the case considered here, the solution for $b_1(\xi)$ is

$$b_1 = (1 - h)a_1 \qquad (37a)$$

where a_1 is given by equation (27). Hence,

$$b_1 = 120(1 - h)/(c^2 - 12c + 60). \qquad (37b)$$

Equation (37a) can be verified by noting that with $A = 0$ and equation (37a), equations (14a) and (14d) imply $F_4 = (1 - h)F_1$. Then, subtracting equation (16) from equation (15) and using this relation yields in this case an equation which, with h constant, is identically satisfied by equation (37a).

With equation (37a) and $A = 0$, it is found that equations (12–13d) imply

$$H/H_1 = h + (1 - h)(u/u_1). \qquad (38)$$

As a check, it is noted that equation (38) is also implied exactly by the original partial differential equations (2) and (6) for flow over a flat plate with $Pr = 1$ and a uniform wall temperature. From equations (38) and (18) it follows that the temperature profiles will be

$$\frac{T}{T_\infty} = h\left[1 + \frac{(\gamma - 1)}{2}M_\infty^2\right] + (1 - h)\left[1 + \left(\frac{\gamma - 1}{2}\right)M_\infty^2\right]\frac{u}{u_\infty} - \left(\frac{\gamma - 1}{2}\right)M_\infty^2\left(\frac{u}{u_\infty}\right)^2. \qquad (39)$$

According to equations (19), (23), (27) and (37) the local skin friction coefficient and Nusselt number can be found from

$$c_f\,Re^{\frac{1}{2}} = 240\,\varphi/[c(c^2 - 12c + 60)] \qquad (40)$$

$$Nu/(c_f Re) = \tfrac{1}{2}. \qquad (41)$$

After $c(\xi)$ has been found, the velocity and temperature profiles in the (ξ, η) plane can be found from equations (13) and (39) with $A = 0$. Profiles in the (x, y) plane can then be found from equations (25) and (10).

It now remains only to find $c(\xi)$ for the general case. For this purpose equation (35) can be written in the form

$$z' = f_1(\xi, z) \tag{42a}$$

where $z = c^2$, or $c = \sqrt{z}$, and

$$f_1(\xi, z) = 2\frac{[\varphi^2(a_1 - c)/K] + [F_1 c^2 \varphi'/\varphi]}{F_1 + c\,\mathrm{d}F_1/\mathrm{d}c}. \tag{42b}$$

Moreover, $a_1(c)$ is given by equation (27), and $F_1(c)$ by equation (14a) with $A = 0$. Equation (42a) is a first-order ordinary differential equation which, for prescribed differentiable non-zero $\varphi(\xi)$, can be solved numerically without difficulty by any of the well-known standard techniques. The initial condition is $\lambda = 0$, and hence (for a finite φ at the leading edge) $z = 0$, at $\xi = 0$.

In most cases, $F_1(c)$ will be virtually constant and hence in equation (42b), F_1 may be replaced by a constant "average" value, with $\mathrm{d}F_1/\mathrm{d}c$ replaced by zero. The calculations then become even simpler.

4. METHOD OF SOLUTION FOR PRESSURE GRADIENT WITH ZERO HEAT TRANSFER. LOCATION OF SEPARATION POINT

In this section the method of solution of the equations here will first be given for the general case of a pressure gradient without heat transfer in which the suction mass flow distribution at the wall is prescribed. Then a mathematically simple class of solutions, namely those for which c = constant, will be discussed. A method for locating more accurately the separation point in an adverse pressure gradient will then be described. Finally, a simplified approximate method of solving the ordinary differential equations here will be shown.

For zero heat transfer and $Pr = 1$, it has already been noted that $h = 1$, while the temperature profiles can be found from the velocity profiles from equation (17). Moreover, since in this case $(\partial T/\partial y)_w = 0$, it follows from equations (18) and (12) that $b_1 = 0$. It is therefore necessary only to solve a single differential equation, namely the momentum equation (8) or (15).

Prescribed distribution of suction velocity

With $\varphi(\xi)$ prescribed, it is first noted from equation (14a) with $b_1 = 0$ and $h = 1$, that $F_1 = F_1(c, a_2)$ while according to equation (13d) $a_2 = a_2(c, A)$. Hence,

$$F_1' = \frac{\partial F_1}{\partial c}c' + \frac{\partial F_1}{\partial a_2}\left(\frac{\partial a_2}{\partial c}c' + \frac{\partial a_2}{\partial A}A'\right). \tag{43}$$

Using equation (13e) for A, equation (15) can then be written in the form

$$c' = N_1/D_1 = f_2(\xi, c) \tag{44a}$$

where

$$\begin{aligned} N_1 = {} & \frac{\varphi^2}{K}\frac{\rho_\infty}{\rho_1}\frac{T_\infty}{T_1}\frac{u_\infty}{u_1}(a_1 - c) \\ & - c^2\left[F_1\left(\frac{T_1'}{T_1} + \frac{\rho_1'}{\rho_1} - \frac{\varphi'}{\varphi}\right)\right. \\ & + \frac{\partial F_1}{\partial a_2}\frac{\partial a_2}{\partial A}\frac{Km}{\varphi^2}\frac{\rho_1}{\rho_\infty}\frac{u_1'}{u_\infty}\left(\frac{u_1''}{u_1'} + \frac{\rho_1'}{\rho_1} - 2\frac{\varphi'}{\varphi}\right) \\ & \left. + \left(F_1 + m\frac{T_\infty}{T_1}F_2\right)\frac{u_1'}{u_1}\right] \end{aligned} \tag{44b}$$

$$\begin{aligned} D_1 = F_1 c + c^2 & \left[\frac{\partial F_1}{\partial c} + \frac{\partial F_1}{\partial a_2}\left(\frac{\partial a_2}{\partial c}\right.\right. \\ & \left.\left. + \frac{2Km}{\varphi^2}\frac{\rho_1}{\rho_\infty}\frac{u_1'}{u_\infty}c\frac{\partial a_2}{\partial A}\right)\right]. \end{aligned} \tag{44c}$$

For a given $u_1/u_\infty(\xi)$, M_∞ and $\varphi(\xi)$, N_1/D_1 is an explicit function of ξ and c. Thus the first-order differential equation (44a) can be solved by any of the standard techniques for such equations. For a sharp leading edge with finite suction the initial condition is $c = 0$ at $\xi = 0$. Consequently, c' will be infinite initially. This difficulty can be overcome by writing equation (44a) in the form

$$\mathrm{d}\xi/\mathrm{d}c = D_1/N_1 = f_3(c, \xi) \tag{44d}$$

and solving this equation for ξ vs. c.†

† Another possibility is to introduce $z = c^2$, $c = \sqrt{z}$, and write equation (44a) in the form $z' = f_4(\xi, z)$. This, however, will introduce square-roots in many places here. In connection with equation (44d), it may, in practice, be found necessary to decrease the increments in c near the end of the calculation.

Class of solutions for which c = constant

The ordinary differential equations here become simplified in the class of cases for which the solutions are such that c = constant. The pressure gradient [or $u_1/u_\infty(\xi)$] is here considered to be prescribed, but the required suction distribution $\varphi(\xi)$ must then be determined. The quantity of suction may be prescribed by prescribing the value of c. This class of solutions is a generalization to non-zero pressure gradients of the case, considered in Section 3 and included in the present class as a special case, of flow over a flat plate with $v_w \sim x^{-\frac{1}{2}}$.

For c = constant, equation (8) with the use of equations (43) and (13e) can be written in the form

$$\lambda' = N_2/D_2 = f_5(\xi, \lambda) \tag{45a}$$

where

$$N_2 = K\frac{\rho_\infty}{\rho_1}\frac{T_1}{T_\infty}\frac{u_\infty}{u_1}(a_1 - c) - \lambda\left\{F_1\frac{\rho_1'}{\rho_1} + \left[F_1 + m\frac{T_\infty}{T_1}F_2\right]\frac{u_1'}{u_1} + \frac{\partial F_1}{\partial a_2}\frac{\partial a_2}{\partial A}\frac{m}{K}\left(\frac{T_\infty}{T_1}\right)^2\frac{\rho_1}{\rho_\infty}\frac{u_1'}{u_\infty}\left[\frac{u_1''}{u_1'} + (3-2\gamma)\frac{\rho_1'}{\rho_1}\right]\lambda\right\} \tag{45b}$$

$$D_2 = \tfrac{1}{2}F_1 + \frac{m}{K}\left(\frac{T_\infty}{T_1}\right)^2\frac{\rho_1}{\rho_\infty}\frac{u_1'}{u_\infty}\frac{\partial F_1}{\partial a_2}\frac{\partial a_2}{\partial A}\lambda. \tag{45c}$$

Again, for a sharp leading edge, $\lambda = 0$ at $\xi = 0$ and equation (45a) can be solved by standard techniques. After $\lambda(\xi)$ has been thus obtained, the suction distribution follows from equation (10):

$$\varphi(\xi) = Kc(T_1/T_\infty)/\sqrt{\lambda}. \tag{46}$$

Location of separation point

The separation point is assumed to occur where $(\partial u/\partial y)_w = 0$, and hence where $a_1 = 0$. With $b_1 = 0$, equations (13b) and (13e) imply that for zero heat-transfer separation occurs where

$$(12 - c)c^2 = -(120/m)(\rho_\infty/\rho_1)(u_\infty/u_1')(1/K)\varphi^2. \tag{47}$$

Thus, if, for example, $\varphi(\xi)$ is prescribed then the separation point according to these equations is found by first obtaining $c(\xi)$ as described above, and then finding the value of ξ for which equation (47) holds. This procedure is based on the use of sixth degree velocity profiles. It has, however, been found [38] that although sixth degree profiles give more accurate results for the location of the separation point than the usual fourth degree profiles, they may still yield appreciable errors. Considerably increased accuracy can be obtained by using for this purpose (and essentially this purpose only) seventh degree profiles [9, 36, 37] satisfying an additional condition at the wall at the separation point. This condition, obtained by differentiating equation (2) twice with respect to η and taking values at $\eta = 0$ at the point where $(\partial u/\partial \eta)_w = 0$, is†

$$\left[\frac{\partial^4}{\partial\eta^4}\left(\frac{u}{u_1}\right)\right]_w = -\left\{c\left[\frac{2}{h}\frac{\partial}{\partial\eta}\left(\frac{H}{H_1}\right) - c\right]\times\frac{\partial^2}{\partial\eta^2}\left(\frac{u}{u_1}\right)\right\}_w. \tag{48}$$

The seventh degree velocity profile satisfying condition (48) in addition to conditions (11a–h) is

$$u/u_1 = \sum_{i=1}^{7} a_i\eta^i \tag{49a}$$

where

$$a_1 = (\tfrac{7}{4}) + [(-\tfrac{1}{2}) + (c/15) + (r/240)]a_2 + (Ab_1)/(30h) \tag{49b}$$

$$a_3 = -(ca_2/3) - (Ab_1)/(6h)$$

$$a_4 = -(r/12)a_2$$

$$a_5 = -(21/4) + [(-\tfrac{5}{2}) + c + (3r/16)]a_2 + (Ab_1)/(2h)$$

$$a_6 = 7 + [3 - (16c/15) - (3r/20)]a_2 - (8Ab_1)/(15h)$$

$$a_7 = -\tfrac{5}{2} + [(-1) + (c/3) + (r/24)]a_2 + (Ab_1)/(6h) \tag{49c}$$

† In obtaining equation (48) it has been assumed that a term proportional to $(\partial u/\partial y)_w(\partial^2 u/\partial x\partial y)_w$ will be negligible close to and at the separation point (cf. the discussion in [38]).

$$a_2 = -\frac{(\frac{7}{2})c + 2A[1 + (b_1 c)/(30h)]}{4 - c + (c^2/120)[16 - c + (2b_1/h)]}. \quad (49d)$$

The stagnation enthalpy profiles remain as in equation (12). Equations (49a–d) are useful not only as a means for locating accurately the separation point, but may be useful also to calculate accurately the velocity profile at that point.

The explicit expressions for F_1 and F_2 now become:

$$\begin{aligned} F_1 = {} & 0{\cdot}1156 + [0{\cdot}00253 - 0{\cdot}000887c \\ & - 0{\cdot}0000850r + (0{\cdot}000287 - 0{\cdot}0000572c \\ & - 0{\cdot}0000043r)(Ab_1/h)]\, a_2 \\ & + [-0{\cdot}001454 + 0{\cdot}000574c \\ & - 0{\cdot}0000572c^2 + 0{\cdot}0000432r \\ & - 0{\cdot}0000085rc - 0{\cdot}0000003r^2]\, a_2^2 \\ & - [0{\cdot}000444 + 0{\cdot}0000143(Ab_1/h)] \\ & \times (Ab_1/h) \end{aligned} \quad (50a)$$

$$\begin{aligned} F_2 = {} & F_1 + \tfrac{1}{2}(1 + h) - 0{\cdot}6875 \\ & + [0{\cdot}02976 - 0{\cdot}00595c - 0{\cdot}000446r]\, a_2 \\ & + [0{\cdot}1071 - 0{\cdot}01190c + 0{\cdot}000595c^2]\, b_1 \\ & - 0{\cdot}002976\,(Ab_1/h). \end{aligned} \quad (50b)$$

The separation point can now be calculated as described previously, except that the immediately-above expressions are now to be used for a_1, a_2, F_1 and F_2. Thus equation (8) must be solved for $\lambda(\xi)$, or equation (15) for $c(\xi)$, using equations (49) and (50), and the separation point is now the point where $a_1 = 0$ according to equation (49b) and hence, in the case of zero heat transfer ($h = 1$, $b_1 = 0$), where

$$c^2[15 - 2c + (c^2/8)] = -(105/m)(\rho_\infty/\rho_1) \times (u_\infty/u_1')(1/K)\varphi^2. \quad (51)$$

Condition (51) replaces condition (47).

Simplified approximate solution of equations and simplified determination of separation point

Although the procedures just shown for solving the ordinary differential equations are essentially straightforward, it is possible to use simplified approximate procedures for solving these equations in the manner to be shown here.

For most cases, it will be found that F_1 remains virtually constant along the flow. Consequently F_1 may be replaced by a constant "average" value, $\bar{F}_1$, with F_1' set equal to zero. Such an approximation has already been made for flows over an impermeable wall [2] and flows with injection [3]. Noting that a_1 is of the form $a_1 = \alpha_1 + \beta_1 A$, and using equations (26), equation (15) can then be written in the form

$$\begin{aligned} z' = {} & \frac{2}{\bar{F}_1}\frac{\varphi^2}{K}\left(\frac{T_1}{T_\infty}\right)^{-\gamma/\gamma-1}\frac{u_\infty}{u_1}(\alpha_1 - \sqrt{z}) \\ & + 2z\frac{\varphi'}{\varphi} + z\frac{u_1'}{u_1}\Bigg[-\frac{2\Phi_1}{\bar{F}_1} \\ & + (\gamma - 1)M_\infty^2\left(\frac{3\gamma - 1}{\gamma - 1} - \frac{\Phi_1}{\bar{F}_1}\right) \\ & \times \left(\frac{T_\infty}{T_1}\right)\left(\frac{u_1}{u_\infty}\right)^2\Bigg] = f_6(\xi, z) \end{aligned} \quad (52)$$

where

$$\begin{aligned} \alpha_1 &= 120/(c^2 - 12c + 60), \\ \Phi_1 &= \bar{F}_1 + F_2 - \beta_1 h \\ \beta_1 h &= [(12 - c)h + b_1]/[c^2 - 12c + 60]. \end{aligned} \quad (53)$$

Equation (52) is considerably simpler than equation (44a).

With $F_1 = \bar{F}_1$, equation (8) similarly reduces to:

$$\begin{aligned} \lambda' = {} & \frac{2K}{\bar{F}_1}\left(\frac{T_1}{T_\infty}\right)^{-\frac{2-\gamma}{\gamma-1}}\frac{u_\infty}{u_1}(\alpha_1 - c) \\ & + \lambda\frac{u_1'}{u_1}\Bigg[-\frac{2\Phi_1}{\bar{F}_1} + (\gamma - 1)M_\infty^2\Bigg(\frac{\gamma + 1}{\gamma - 1} \\ & - \frac{\Phi_1}{\bar{F}_1}\Bigg)\frac{T_\infty}{T_1}\left(\frac{u_1}{u_\infty}\right)^2\Bigg]. \end{aligned} \quad (54)$$

Equation (54) is useful, for example, for cases of an impermeable wall ($c = 0$) and for the class of solutions for which $c =$ constant. In connection with the latter class it is seen that equation (54) is considerably simpler than equation (45a).

In addition to setting F_1 = constant, there is a possible further simplification, namely to also set F_2 = constant. Such an approximation has been made to good advantage for an impermeable wall [2] and for a transpiration-cooled wall [3]. However, although F_2 does vary to only a limited extent, it will be found that F_2 will vary appreciably more than F_1 in cases of suction. Moreover, replacing F_2 by a constant ("average") value in equation (52) will not greatly simplify the equation. It is noteworthy, on the other hand, that for the special class of solutions for which c = constant the replacement of F_2 by a constant value $\bar{F}_2$ will make equation (54) linear in λ and hence will yield the following relatively simple closed-form solution:

$$\lambda = \frac{2K}{\bar{F}_1}(\alpha_1 - c)\frac{\int_0^\xi \left(\frac{u_1}{u_\infty}\right)^{\frac{2\bar{\Phi}_1}{\bar{F}_1} - 1} \left(\frac{T_1}{T_\infty}\right)^{\frac{2\gamma - 1}{\gamma - 1} - \frac{\bar{\Phi}_1}{\bar{F}_1}} d\xi}{\left(\frac{u_1}{u_\infty}\right)^{\frac{2\bar{\Phi}_1}{\bar{F}_1}} \left(\frac{T_1}{T_\infty}\right)^{\frac{\gamma + 1}{\gamma - 1} - \frac{\bar{\Phi}_1}{\bar{F}_1}}} \tag{55}$$

where $\bar{\Phi}_1 = \bar{F}_1 + \bar{F}_2 - \beta_1 h$. Although equation (55) will be subject to some quantitative errors, it should be noted that equations (55) and (46) afford a rather simple class of approximate solutions for compressible flows with zero heat transfer in an (essentially arbitrarily prescribed) pressure gradient with (an implicitly determined distribution of) suction.

The foregoing simplified procedures apply for calculating properties of the boundary layer prior to separation, and are based on the sixth degree velocity profiles. For a more accurate location of the separation point based on seventh degree velocity profiles, the above types of approximation can again be fruitfully made. In particular F_1 may be replaced by a constant value F_{1s}, its value at the separation point.† Such a procedure has already been found to lead to accurate results for an impermeable wall [9, 36]. With F_1 replaced by F_{1s}, equation (52) remains valid, but with $\bar{F}_1$ replaced by F_{1s} and with α_1, β_1, Φ_1 replaced, respectively, by α_2, β_2, and Φ_2, where

$$\alpha_2 = 7/\{4 - c + (c^2/120)[16 - c + (2b_1/h)]\}, \qquad \Phi_2 = F_{1s} + F_2 - \beta_2 h$$
$$\beta_2 h = \{h - (c/120)[(16 - c)h + 2b_1] + (2b_1/15)\}/\{4 - c + (c^2/120)[16 - c + (2b_1/h)]\}. \tag{56}$$

Thus equation (52) in conjunction with equations (56) must now be solved for $z(\xi)$, whence $c(\xi) = \sqrt{z}$ is found, and the separation point obtained by finding the value of ξ for which equation (51) is satisfied. The calculations will be comparatively simple and straightforward. A basically similar simplified procedure for determining the separation point for the constant-c class of solutions can be readily developed.

As in the calculation of properties before separation it is possible, in calculating the separation point by the seventh degree profiles, to add the simplifying approximation of replacing F_2 by F_{2s}, its value at the separation point. As explained previously, such an additional approximation appears of doubtful worth for flows in which the suction distribution is prescribed. For the constant-c class of flows, however, it is again noted that with such an additional simplification, the closed-form solution for $\lambda(\xi)$ given by equation (55) will hold, but with α_1, β_1, $\bar{\Phi}_1$ replaced by α_2, β_2, $\bar{\Phi}_2$.

In general, after the solutions (whether for separation or prior to separation) have been obtained by the simplified approximate procedures described here, an indication of their

† A rough value, c_s, of c at the separation point may in general be obtained in advance by estimating a rough value, ξ_s, of ξ at which separation may occur, evaluating the right side of equation (51) at this ξ, and then solving for c. This may then be used to evaluate F_{1s}, which will be relatively insensitive to the values of c_s and ξ_s.

accuracy or reliability can be obtained by calculating F_1 (and, if pertinent, F_2) with these solutions and seeing how widely it actually varies.†

5. PRESSURE GRADIENT WITH HEAT TRANSFER

The comparatively general case considered in this section is that of a prescribed uniform wall temperature in a given pressure gradient with suction. In this case a constant value of h (not necessarily one) is prescribed. The equations for this case are now considerably more complicated because of the appearance of an additional unknown, $b_1(\xi)$. Equations (15) and (16) must now be solved simultaneously for $\lambda(\xi)$ and $b_1(\xi)$. The present study will be confined to indicating and applying a very approximate, but comparatively simple, method of solution of the equations with heat transfer. The method will be similar to that shown and applied for an impermeable wall [2, 35] and a transpiration-cooled wall [3].

In equations (15) F_1 may still be replaced by a constant value $\bar{F}_1$, with $F_1' = 0$. Moreover, in F_2 and a_1 (as well as in F_1 and a_2) b_1 may at first be replaced by an approximate value $\bar{b}_1$. In the case of a sharp leading edge the value for $\bar{b}_1$ here might be taken to be the expression for b_1 for flow over a flat plate, as given by equation (37b). Equation (15) then becomes uncoupled from equation (16) and can be solved separately. In fact, equation (15) is reduced to equation (52), and the numerical solution of this equation then proceeds quite similarly to the case of zero heat transfer, except that one must now take into account the b_1 terms in Φ_1, and insert the given value of h (instead of $h = 1$) wherever it appears. To solve equation (16) approximately for $b_1(\xi)$, set $F_4' = 0$ with F_4 replaced by a constant "average" value $\bar{F}_4$. Subtracting equation (16) from equation (15) the following equation is then obtained:

$$(F_2 - \beta_1 h)(A/h) = \alpha_1 - c - [b_1 - (1-h)c] \times (\bar{F}_1/\bar{F}_4). \tag{57}$$

With equations (14b) and (53) for F_2, β_1, and α_1, equation (57) is a linear equation in b_1 whose solution is

$$b_1(\xi) = N_3/D_3 \tag{58}$$

where

$$N_3 = \left(\frac{120}{c^2 - 12c + 60}\right) + c\left[(1-h)\frac{\bar{F}_1}{\bar{F}_4} - 1\right] - \frac{A}{h}\left[\bar{F}_1 + \frac{1}{2}(1+h) - 0{\cdot}7143 - \frac{30(0{\cdot}01905 - 0{\cdot}00238c)(2c + A) + (12-c)h}{c^2 - 12c + 60}\right]$$

$$D_3 = \frac{\bar{F}_1}{\bar{F}_4} + \frac{A}{h}\left[0{\cdot}1071 - 0{\cdot}01191c + 0{\cdot}000595c^2 - 0{\cdot}00119\frac{A}{h} - \frac{(0{\cdot}01905 - 0{\cdot}00238c)(Ac/2h) + 1}{c^2 - 12c + 60}\right].$$

Equation (58) may be used to obtain (at least roughly) heat-transfer properties.

The separation point can be located in a manner quite similar to the simplified method described for zero heat transfer. Equation (52) must be solved for $z(\xi)$ inserting now the given value of h in the expression for α_2 and Φ_2 as given by equations (56) and (50b), and also inserting as an approximation for b_1 its value for a flat plate (assuming a sharp leading edge) as given by equation (37b). A further simplification can be made by replacing F_2 here by a constant value F_{2s}, its (estimated) value at $c = c_s$. After $c(\xi)$ has been thus obtained the separation point can be found approximately by determining the value of ξ at which a_1 as given by equations (49b) and (49c) [with b_1 there given by equation (37b)] vanishes.

The procedures outlined here can also be

† It should be kept in mind that the method described here of determining the separation point by the use of *seventh* degree velocity profiles is predicated on the assumption that separation will indeed occur. cf. the example in Section 7.

readily adapted to the constant-c class of solutions with heat transfer.

6. ASYMPTOTIC SUCTION SOLUTIONS

It was shown in [24] that under quite general conditions, in the limiting case of infinite suction, i.e. $\varphi \to \infty$, the exact solutions of the original partial differential equations for u/u_1 and T/T_1 as functions of ζ and M_1 approach the asymptotic suction solutions for flow over a flat plate. It will be shown in this section that such will also be the case with the equations used here. Since it has already been shown that in the asymptotic case of $\varphi \to \infty$ the equations used here yield results in good agreement with the exact solutions for incompressible flow over a flat plate, and since the effect of compressibility for flow over a flat plate according to the equations used here is in accord with exact solutions, it follows that the basic equations used here, namely equations (15) and (16) in conjunction with the sixth degree velocity profiles and the seventh degree stagnation enthalpy profiles, will yield results in satisfactory agreement with exact solutions for compressible flows with pressure gradient (and heat transfer) when the suction parameter φ is large. This is, of course, an important further check on the accuracy of the equations used here.

To show that according to the equations used here the solutions for the velocity and temperature profiles (in terms of ζ and M_1) will approach those for flow in a zero pressure gradient when $\varphi \to \infty$, consider a solution in which, as $\varphi \to \infty$, $\lambda \to 0$ and hence by equation (13e) $A \to 0$. It is then seen from equations (14a–d) that in this limit the expressions for F_1 to F_5 in terms of c and b_1 remain the same as for flows without a pressure gradient. Moreover, if one brings the braced term in equation (15) to the right side, and considers $\varphi \to \infty$ with c remaining finite and therefore c' approaching zero, then the equation obtained in this limit is that which will result by setting the new entire right side of equation (15) equal to zero. However, $\varphi \to \infty$, the φ^2 term there will predominate, and hence in the limit this right side will vanish if and only if $(a_1 - c) = 0$. Using equation (13b) with $A = 0$, this will lead to exactly the same (finite) value of c as for the asymptotic suction solution for flow over a flat plate. It then follows from equations (13a–e) that the velocity profiles u/u_1 will be the same functions of η as for the flat plate flows. By similar reasoning it is seen that equation (16) in the asymptotic case reduces to the equation $b_1 = (1 - h)\,c$, which is the same as the equation which would be obtained for flow over a flat plate. Thus, the basic equations used here imply that as $\varphi \to \infty$ the velocity and stagnation enthalpy profiles u/u_1 and H/H_1 will tend to the same functions of η as for flow without a pressure gradient. In particular, equation (38) will hold. Now, according to an exact analysis [24] u/u_1 must remain the same function of ζ as for flows without a pressure gradient, where

$$\zeta = -\rho_w v_w \int_0^y (\mathrm{d}y/\mu).$$

If μ is assumed to vary according to equation (5), then it is found from the definitions that $\eta = (1/c)\,\zeta$. Hence the result obtained here for the velocity profiles is in accord with the exact asymptotic solutions. Moreover, equation (18) here, and equations (11) and (15) of [24], imply that for $Pr = 1$ the exact asymptotic temperature profiles will be such that equation (38) will hold, and this is also the case with the approximate equations used here. It may be noted that the entire discussion here holds for a variable, as well as a uniform, suction parameter $\varphi(\xi)$.

An important corollary of the fact that the velocity profiles u/u_1 (in the variable ζ) in general tend to the flat plate profiles as $\varphi \to \infty$ is that in any given (finite) adverse pressure gradient it should always be possible to entirely prevent separation by a sufficient amount of suction.† This statement holds for the general

† A quite recent illustration of this is afforded by the similarity solutions of Zamir and Young [41].

case of compressible flows with or without heat transfer.

7. FLOW WITH A LINEARLY DIMINISHING EXTERNAL VELOCITY

To see explicitly the nature of the solutions and the effect of suction in an adverse pressure gradient, the equations and procedures developed here have been applied to the case of an external flow characterized by

$$u_1/u_\infty = 1 - \xi. \tag{59}$$

Solutions for constant c

As already explained, the class of solutions for which c = constant are relatively easy to obtain, but the corresponding suction distribution is determined, instead of prescribed. For an external flow given by equation (59) and zero heat transfer, equation (45a) has been solved for various constant c for $0 < c \leqslant 3$. The results obtained for the momentum thickness

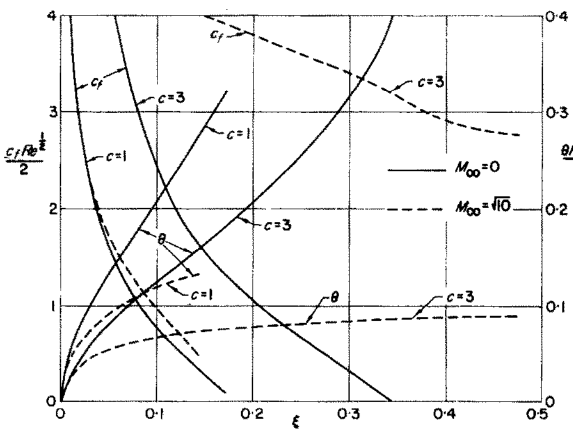

FIG. 4. Distribution of momentum thickness and skin-friction for constant-c solutions. $u_1/u_\infty = 1 - \xi$; zero heat transfer.

and skin-friction distribution are shown in Fig. 4 for $M_\infty = 0$ and $M_\infty = \sqrt{10}$. The corresponding suction distributions are shown in Fig. 5. In addition, the separation points have been calculated on the basis of the seventh-degree velocity profiles, with the results: $\xi_s = 0{\cdot}173$ and $0{\cdot}370$† for $c = 1$ and $c = 3$, respectively, when $M_\infty = 0$, while $\xi_s = 0{\cdot}145$ for $c = 1$ and $M_\infty = \sqrt{10}$. For $c = 3$ and $M_\infty = \sqrt{10}$, separation apparently does not occur (due to the large suction associated with this case, cf. Fig. 5). As a check on the simplified

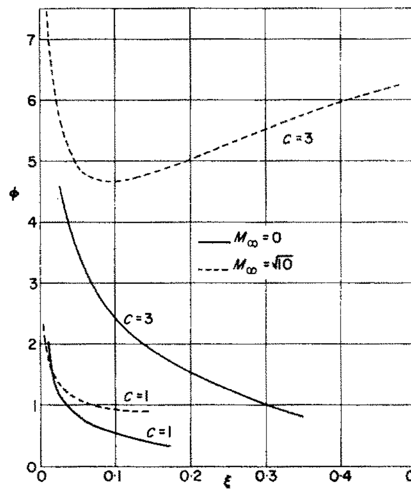

FIG. 5. Distribution of mass flow parameter $\varphi(\xi)$ for "constant-c" solutions.

method based on a constant F_1, equation (54) was solved for these cases, and the results were found to remain very close to those shown here.

Solutions for uniform suction and incompressible flow. Separation. Critical suction velocity

The results obtained here for φ = constant are of considerable physical interest, and will be discussed in some detail.

First, solutions obtained for incompressible flow for the displacement and momentum thickness, and the skin-friction distributions are shown in Figs. 6 and 7 for various values of φ. For this purpose equation (44d) was first solved for $\varphi = 0{\cdot}1$ and $0{\cdot}5$, and then equation (52), based on the simplifying approximation of a constant F_1 along the entire flow, was solved. The results were found to be practically identical. In all of the remaining calculations the procedure based on a constant F_1 was used.

† This is the one case encountered herein in which the seventh-degree profiles lead to a later separation point than the sixth-degree profiles.

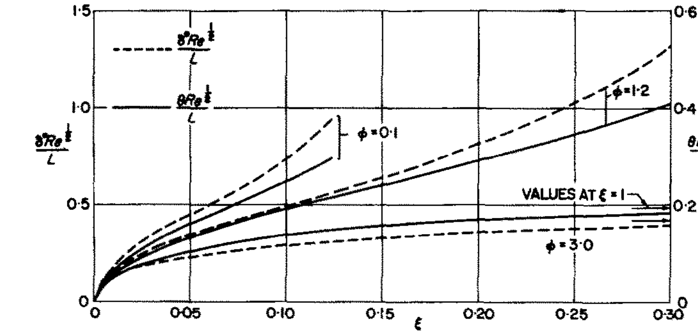

FIG. 6. Distribution of displacement and momentum thickness for uniform suction parameter φ, $u_1/u_\infty = 1 - \xi$, incompressible flow ($M_\infty = 0, h = 1$).

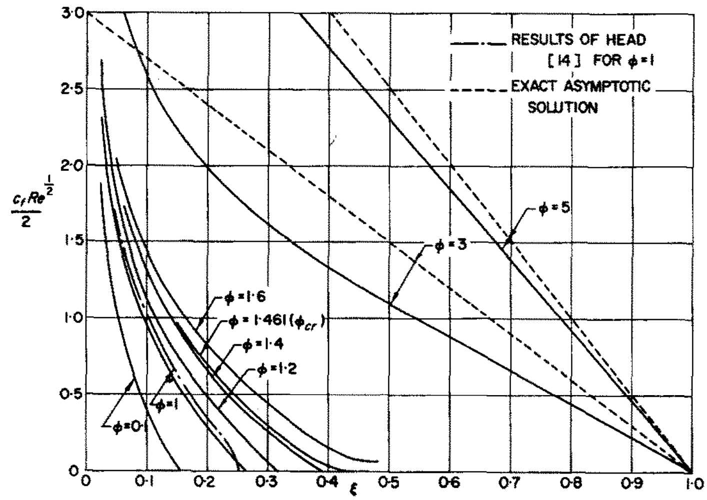

FIG. 7. Distribution of skin friction coefficient for various values of the suction parameter. $u_1/u_\infty = 1 - \xi$, incompressible flow without heat transfer.

In the case of incompressible flow with $u_1/u_\infty = 1 - \xi$ and constant φ, equation (52) can be solved by separation of variables with the result

$$\xi = 1 - \exp\left[- \bar{F}_1 \int_0^c f(c)\,\mathrm{d}c\right] \tag{60a}$$

where

$$f(c) = c/[\varphi^2(\alpha_1 - c) + c^2\Phi_1]. \tag{60b}$$

The results of Head [14] for c_f, based on his momentum-and-energy-integral method, for the case $\varphi = 1$ are included in Fig. 7, and these agree rather well with the results obtained here.

The results shown here for the skin-friction are of especial interest. It is seen that according to Fig. 7, separation will occur at some point if the suction is sufficiently small so that $\varphi \leqslant 1{\cdot}461$. At $\varphi = 1{\cdot}461$, in fact, it is seen that at the separation point, $\mathrm{d}c_f/\mathrm{d}\xi = 0$.† For $\varphi > 1{\cdot}461$ it is found that separation will not occur anywhere. Thus, the value $\varphi = 1{\cdot}461$, to be denoted by φ_{cr}, may be regarded here as the "critical" value of the suction parameter φ.

† This is obtained from the method of analysis used here. The actual behavior of the derivative $\mathrm{d}c_f/\mathrm{d}\xi$ for $\varphi = \varphi_{cr}$ *at*, and in a small region *near* $\xi = \xi_s$ according to an *exact* solution may be worthy of investigation.

This result can be obtained analytically by noting that according to equation (47) separation will occur in the present incompressible flow where $c = c_s$, and

$$c_s^3 - 12c_s^2 + 120\varphi^2 = 0. \qquad (61)$$

c_s vs. φ according to equation (61) is plotted in Fig. 8, and it is seen that there are no (physically

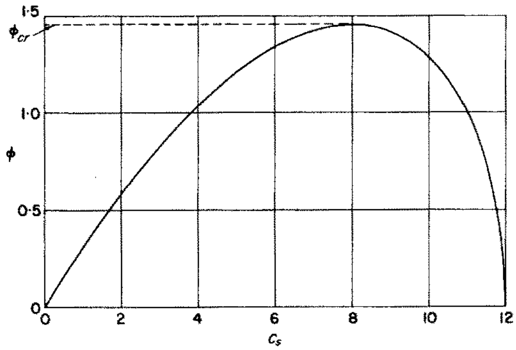

FIG. 8. Variation of c_s with suction parameter φ, for $u_1/u\infty = 1 - \xi$, incompressible flow and zero heat transfer.

significant) roots for c_s when $\varphi > 1{\cdot}461$. At $\varphi = 1{\cdot}461 = \varphi_{cr}$, it is seen that $dc_s/d\varphi \to \infty$, or $d\varphi/dc_s = 0$. The location of the separation point ξ_s according to equations (60) and (61) is shown in Fig. 9, together with results of Thwaites [4] and of Curle [15]. (Thwaites was able to obtain results only for $\varphi \leqslant 0{\cdot}833$.) Although, as previously noted, the results for ξ_s obtained by equations (60) and (61), based on sixth degree velocity profiles, are subject to some quantitative inaccuracy they are of importance in indicating the existence of a critical φ and in determining its approximate value. The existence of such a φ has already been predicted (Section 6) here on the basis of the general asymptotic suction solutions.

It should be noted that the results for ξ_s vs. φ given by Curle in [15] and shown in Fig. 9 were based on setting $c_f = 0$ in equation (23) of [15] and solving for ξ. However, further study of this equation, for the purpose of calculating the skin-friction distribution therefrom,† has indicated that for a fixed φ the curve of c_f vs. ξ will consist of more than one branch (since it is of fourth degree), and that for the higher values of the suction parameter (including $\varphi \geqslant 1{\cdot}2$) the branch which contains the point for which $c_f = 0$ is a branch approaching the ξ axis from below, i.e. from negative c_f, and hence from artificial values. The branch of positive c_f in these cases does not cross the ξ axis. Thus the curve of Curle near and at $\varphi = \varphi_{cr}$ does not appear to have an entirely consistent physical basis, but there is a nevertheless noteworthy similarity between Curle's curve and that obtained by the present procedure.

From Fig. 9 it is seen that according to the results obtained here, ξ_s has a definite value $(= 0{\cdot}438) < 1$ for $\varphi = \varphi_{cr}$. When $\varphi > \varphi_{cr}$, separation does not occur anywhere.

To obtain quantitatively more accurate results for ξ_s vs. φ when $\varphi < \varphi_{cr}$, the procedure based on the seventh degree velocity profiles has been applied to the flows considered here. The results are shown in Fig. 9. The calculations were based on F_1 = constant, but calculations for $\varphi \leqslant 1{\cdot}5$ without this assumption (see Fig. 9) gave very similar results. It should be noted that the one value of ξ_s, namely for $\varphi = 1$, calculated by Head [14] by his method practically coincides with that obtained here. The results based on the seventh degree profiles are qualitatively similar to those based on the sixth degree profiles for $0 \leqslant \varphi \leqslant 1{\cdot}3$ (roughly), but deviate considerably for larger φ and do not yield a critical φ in the manner described above. For $\varphi > 2{\cdot}21$, however, it is found that a single-valued solution for c vs. ξ cannot be obtained for all ξ before any separation. This is probably due to the inaccuracy in the seventh-degree profiles away from the separation point, due to neglecting a term proportional to $\tau\partial\tau/\partial x$ in the use of these profiles. The exact curve of ξ_s vs. φ may be expected to follow the seventh degree curve in Fig. 9 at least for lower φ's.

† The authors acknowledge the aid of Mr. Denis Blackmore in these calculations.

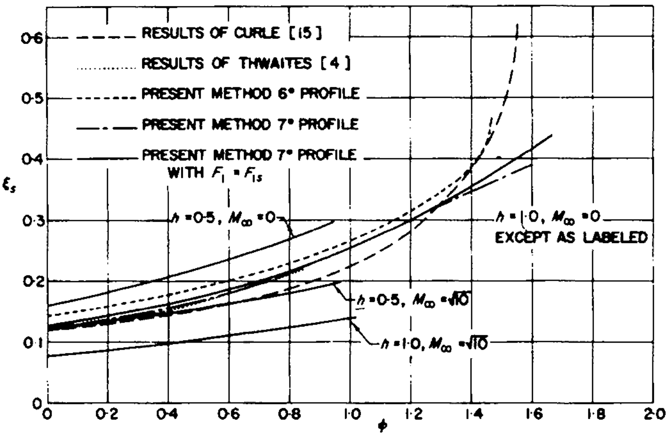

FIG. 9. Separation point vs. suction parameter for $u_1/u_\infty = 1 - \xi$.

For higher φ's the exact curve may be expected to appear like the sixth degree curve of Fig. 9, although further research may be warranted to verify this.†

Approach to asymptotic suction profiles

To note the approach of the solutions to the general asymptotic suction profiles, solutions for incompressible flow with $u_1/u_\infty = 1 - \xi$ and constant suction were obtained for $\varphi = 3$ and $\varphi = 5$. According to equation (19) of [24], the asymptotic suction solution for these flows will yield

$$(c_f)_{\varphi\to\infty} = 2\varphi(1 - \xi)\,Re^{-\frac{1}{2}}. \qquad (62)$$

It should be noted that equation (62) is also implied exactly by equation (19) of the present analysis, since for $\varphi \to \infty$, $a_1 - c = 0$. Figure 7 illustrates, for $\varphi = 3$ and 5, the approach of the present solutions for the skin friction to the asymptotic suction solutions. Moreover, to contrast the nature of the solutions for $\varphi < \varphi_{cr}$ and $\varphi > \varphi_{cr}$ the development of the velocity profiles along the flow is shown in Fig. 10 for $\varphi = 0.8$, 3 and 5. The asymptotic suction profile ($\varphi \to \infty$) is also shown here, and the approach toward this profile for $\varphi = 5$ can be clearly seen.

† An early problem of Prandtl's [39] and considered, for example, recently by Head [13] and by Curle [15] has been to find the required suction distribution to maintain a zero skin-friction layer, in a flow characterized by equation (59), starting at the point ($\xi = 0{\cdot}120$) at which $(\partial u/\partial y)_w$ first vanishes. This essentially initial-value problem is different from that considered here.

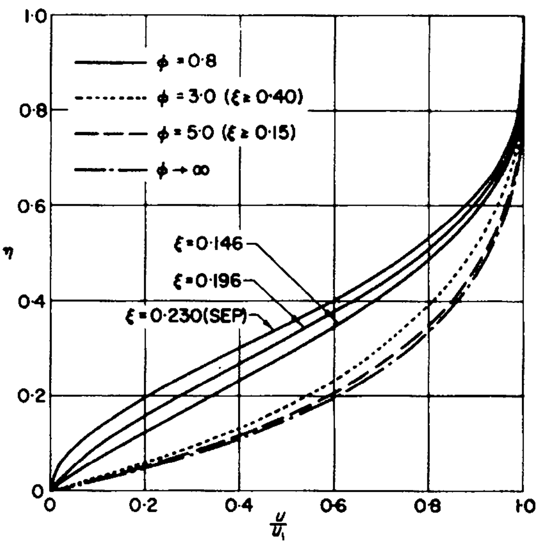

FIG. 10. Velocity profiles for $u_1/u_\infty = 1 - \xi$ incompressible flow without heat transfer.

Mach number and heat-transfer effects

An indication of the effect of compressibility, with uniform suction, can be seen in Fig. 9, which shows the results obtained here (by the seventh-degree-velocity-profile method) for the separation point vs. φ for $M_\infty = 0$ and $M_\infty \sqrt{10}$ with zero heat transfer ($h = 1$) and with the wall cooled ($h = 0{\cdot}5$).† The forward movement of the separation point with Mach number for a fixed h may be noted here, together with the delay of separation due to wall cooling. Moreover, the minimum ("critical") values of φ above which separation is entirely avoided with zero heat transfer was calculated based on sixth degree profiles for $M_\infty = 0$ to $\sqrt{10}$, and the results are shown in Fig. 11. φ_{cr} is here seen to increase fairly rapidly with Mach number.

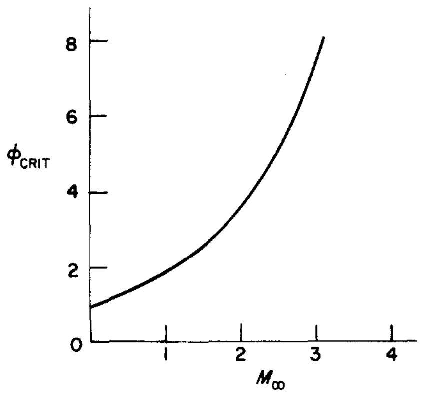

FIG. 11. Critical suction parameter for prevention of separation. Zero heat transfer. $u_1/u_\infty = 1 - \xi$.

For the low-speed ($M_\infty = 0$) flow characterized by equation (59) with uniform suction it may be noted that (admittedly very) approximate values of φ_{cr} with heat transfer can be quite easily obtained from the equations here, similarly to the manner described above for zero heat transfer ($h = 1$). Thus, setting $a_1 = 0$, where a_1 is given by equations (13b) and (13e), yields (for $u_1/u_\infty = 1 - \xi$ and $M_\infty = 0$)

$$120 - hc_s^2(K/\varphi^2)[12 - c_s + (b_1/h)] = 0. \qquad (63)$$

Approximating b_1 by equation (37b) (with c replaced by c_s) and determining the value of φ for a given h for which $d\varphi/dc_s = 0$ yields the results for φ_{cr} vs. h shown in Fig. 12. Figure 12 shows that for the range calculated ($0{\cdot}3 \leqslant h \leqslant 1{\cdot}4$) the effect of wall heating or cooling on φ_{cr} is surprisingly small and in fact in the unexpected direction, in spite of the fact that wall cooling with a given suction parameter φ does appreciably delay separation (Fig. 9). Further investigation of this problem, for example a precise determination, if possible, of φ_{cr} vs. h, would appear worthwhile.

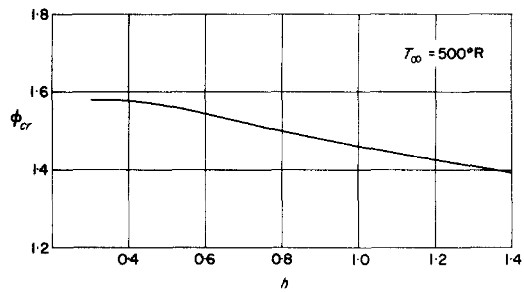

FIG. 12. Variation of critical suction parameter with h for $u_1/u_\infty = 1 - \xi$, $M_\infty = 0$.

† For $h = 0{\cdot}5$ the simplifying approximation of a constant F_2, viz. $F_2 = F_{2s}$ (in addition to a constant F_1) was made.

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Résumé—La couche limite laminaire compressible avec gradient de pression et aspiration est analysée sur la base des équations intégrales de quantité de mouvement et de l'énergie en utilisant des profils de vitesse du sixième et (pour le décollement) du septième degré et des profils d'enthalpie d'arrêt du septième degré. Pour les écoulements sur une plaque plane et avec un gradient de pression, des méthodes directes et simples sont données pour calculer la couche limite avec un nombre de Mach donné, une température pariétale uniforme donnée et une distribution donnée d'aspiration. Les résultats obtenus par la théorie actuelle sont en bon accord avec des solutions disponibles exactes ou prétendant être précises, dans le cas d'une paroi perméable ou non, comprenant les solutions générales asymptotiques avec aspiration pour des écoulements compressibles avec gradient de pression et transport de chaleur. Les solutions asymptotiques sembleraient montrer qu'il serait toujours possible d'éviter complètement le décollement avec un gradient de pression contraire donné (fini) à l'aide d'une aspiration suffisante. On expose un cas mathématiquement simple de solutions pour lesquelles le gradient de pression est fixé arbitrairement, mais la distribution d'aspiration est déterminée de façon implicite. Finalement, la couche limite avec une vitesse extérieure diminuant linéairement, est calculée en détail. Ce qui est spécialement intéressant est le retard et la disparition du décollement par l'aspiration, y compris la détermination du minimum du paramètre d'aspiration (homogène) nécessaire pour éviter entièrement le décollement. Les effets du nombre de Mach et de la température pariétale sont exposés.

Zusammenfassung—Die kompressible laminare Grenzschicht mit Druckgradienten und Absaugung wird analysiert auf Grund der Integralgleichungen für Impuls und Energie und in Verbindung mit Geschwindigkeitsprofilen sechster und (für Ablösung) siebter Ordnung und Profilen für die Staupunktenthalpie von siebter Ordnung. Für Strömungen entlang ebener Platten und Strömungen mit Druckgradienten werden direkte und einfache Methoden zur Berechnung der Grenzschicht bei gegebener Machzahl, gegebener gleichförmiger Wandtemperatur und gegebener Absaugungsverteilung mitgeteilt. Die in der vorliegenden Analyse erhaltenen Ergebnisse stimmen gut mit verfügbaren exakten oder ziemlich genauen Lösungen für durchlässige und undurchlässige Wände überein, einschliesslich der allgemeinen assymptotischen Absaugungslösungen für kompressible Strömungen mit Druckgradienten und Wärmeübergang. Die assymptotischen Lösungen deuten an, dass es immer möglich sein müsste, durch ausreichende Absaugung eine Ablösung bei gegebenem (endlichem) gegenläufigen Druckgradienten vollständig zu verhindern. Eine mathematisch einfache Lösung wird angegeben für einen beliebig vorgeschriebenen Druckgradienten bei einer implizit bestimmten Absaugungsverteilung. Schliesslich wird die Grenzschicht für linear abnehmende äussere Geschwindigkeit im einzelnen berechnet. Von besonderem Interesse hier ist die Verzögerung und vollständige Vermeidung einer Ablösung durch Absaugung, wofür die kleinsten (homogenen) Saugparameter bestimmt werden. Einflüsse der Machzahl und der Wandtemperatur sind angegeben.

Аннотация—Проводится анализ сжимаемого ламинарного пограничного слоя с градиентом давления при наличии отсоса на основе интегральных уравнений момента и тепла в сочетании с распределением скорости в шестой и седьмой (для отрыва) степени и распределения энтальпии торможения в седьмой степени.

Показана возможность применения перспективных и простых методов решения пограничного слоя для случаев обтекания плоской пластины и течений при наличии градиента давления при заданных числах Маха, заданном равномерном распределении температуры стенки и заданном распределении отсоса. Полученные результаты хорошо согласуются с известными точными или практически точными решениями для непроницаемой или проницаемой стенки, включая общие асимптотические решения для отсоса в потоках сжимаемой жидкости при наличии теплообмена и градиента давления. Асимптотические решения показывают, что при достаточном отсосе всегда можно полностью устранить отрыв при заданном (конечном) отрицательном градиенте давления. Рассмотрен математически простой случай при произвольно заданном градиенте давления и определяемом в неявном виде распределении отсоса. Наконец, сделаны подробные расчеты пограничного слоя при линейно уменьшающейся внешней скорости. Особый интерес представляет задержка и полное предотвращение отрыва при отсосе, включая определения минимального (гомогенного) параметра отсоса для полного предотвращения отрыва. Учитывается влияние числа Маха и температуры стенки.